

# DIFFEOMORPHIC APPROXIMATION OF SOBOLEV HOMEOMORPHISMS

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ABSTRACT. Every homeomorphism  $h: \mathbb{X} \rightarrow \mathbb{Y}$  between planar open sets that belongs to the Sobolev class  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ ,  $1 < p < \infty$ , can be approximated in the Sobolev norm by  $\mathcal{C}^\infty$ -smooth diffeomorphisms.

## 1. INTRODUCTION

By the very definition, the Sobolev space  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{R})$ ,  $1 \leq p < \infty$ , in a domain  $\mathbb{X} \subset \mathbb{R}^n$ , is the completion of  $\mathcal{C}^\infty$ -smooth real functions having finite Sobolev norm

$$\|u\|_{\mathcal{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathcal{L}^p(\mathbb{X})} + \|\nabla u\|_{\mathcal{L}^p(\mathbb{X})} < \infty.$$

The question of smooth approximation becomes more intricate for Sobolev mappings, whose target is not a linear space, say a smooth manifold [11, 19, 20, 21] or even for mappings between open subsets  $\mathbb{X}, \mathbb{Y}$  of the Euclidean space  $\mathbb{R}^n$ . If a given homeomorphism  $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$  is in the Sobolev class  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$  it is not obvious at all as to whether one can preserve injectivity property of the  $\mathcal{C}^\infty$ -smooth approximating mappings. It is rather surprising that this question remained unanswered after the global invertibility of Sobolev mappings became an issue in nonlinear elasticity [4, 17, 31, 35]. It was formulated and promoted by John M. Ball in the following form.

**Question.** [6, 7] If  $h \in \mathcal{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$  is invertible, can  $h$  be approximated in  $\mathcal{W}^{1,p}$  by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans and points out its relevance to the regularity of minimizers of neo-hookean energy functionals [5, 9, 14, 16, 34]. Partial results toward the Ball-Evans problem were obtained in [30] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [10] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [6, 33] illustrate the difficulty of preserving invertibility in the approximation process. In [24] we provided an affirmative answer to the Ball-Evans question in the planar case when  $p = 2$ . In the present

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paper we extend the result of [24] to all Sobolev classes  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$  with  $1 < p < \infty$ . The case  $p = 1$  still remains open.

Let  $\mathbb{X}$  be a nonempty open set in  $\mathbb{R}^2$ . We study complex-valued functions  $h = u + iv: \mathbb{X} \rightarrow \mathbb{C} \simeq \mathbb{R}^2$  of Sobolev class  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ ,  $1 < p < \infty$ . Their real and imaginary part have well defined gradient in  $\mathcal{L}^p(\mathbb{X}, \mathbb{R}^2)$

$$\nabla u: \mathbb{X} \rightarrow \mathbb{R}^2 \quad \text{and} \quad \nabla v: \mathbb{X} \rightarrow \mathbb{R}^2.$$

Then we introduce the gradient mapping of  $h$ , by setting

$$(1.1) \quad \nabla h = (\nabla u, \nabla v): \mathbb{X} \rightarrow \mathbb{R}^2 \times \mathbb{R}^2.$$

The  $\mathcal{L}^p$ -norm of the gradient mapping and the  $p$ -energy of  $h$  are defined by

$$(1.2) \quad \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})} = \left[ \int_{\mathbb{X}} (|\nabla u|^p + |\nabla v|^p) \right]^{\frac{1}{p}}, \quad \mathbf{E}_{\mathbb{X}}[h] = \mathbf{E}_{\mathbb{X}}^p[h] = \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})}^p.$$

The reader may wish to notice that this norm is slightly different from what can be found in other texts in which the authors use the differential matrix of  $h$  instead of the gradient mapping, so

$$(1.3) \quad \|Dh\|_{\mathcal{L}^p(\mathbb{X})} = \left[ \int_{\mathbb{X}} (|\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \right]^{\frac{1}{p}}.$$

Thus our approach involves *coordinate-wise*  $p$ -harmonic mappings, which we still call  $p$ -harmonic for the sake of brevity. We shall take an advantage of the gradient mapping on numerous occasions, by exploring the associated *uncoupled* system of real  $p$ -harmonic equations for mappings with smallest  $p$ -energy. Our theorem reads as follows.

**Theorem 1.1.** *Let  $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$  be an orientation-preserving homeomorphism in the Sobolev space  $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$ ,  $1 < p < \infty$ , defined for open sets  $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ . Then there exist  $\mathcal{C}^\infty$ -diffeomorphisms  $h_\ell: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ,  $\ell = 1, 2, \dots$  such that*

- (i)  $h_\ell - h \in \mathcal{W}_o^{1,p}(\mathbb{X}, \mathbb{R}^2)$ ,  $\ell = 1, 2, \dots$
- (ii)  $\lim_{\ell \rightarrow \infty} (h_\ell - h) = 0$ , uniformly on  $\mathbb{X}$
- (iii)  $\lim_{\ell \rightarrow \infty} \|\nabla h_\ell - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} = 0$
- (iv)  $\|\nabla h_\ell\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})}$ , for  $\ell = 1, 2, \dots$
- (v) If  $h$  is a  $\mathcal{C}^\infty$ -diffeomorphism outside of a compact subset of  $\mathbb{X}$ , then there is a compact subset of  $\mathbb{X}$  outside which  $h_\ell \equiv h$ , for all  $\ell = 1, 2, \dots$

A straightforward triangulation argument yields the following corollary.

**Corollary 1.2.** *Let  $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$  be an orientation-preserving homeomorphism in the Sobolev space  $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$ ,  $1 < p < \infty$ , defined for open sets  $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ . Then there exist piecewise affine homeomorphisms  $h_\ell: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ,  $\ell = 1, 2, \dots$  such that*

- (i)  $h_\ell - h \in \mathcal{W}_o^{1,p}(\mathbb{X}, \mathbb{R}^2)$ ,  $\ell = 1, 2, \dots$

- (ii)  $\lim_{\ell \rightarrow \infty} (h_\ell - h) = 0$ , uniformly on  $\mathbb{X}$
- (iii)  $\lim_{\ell \rightarrow \infty} \|\nabla h_\ell - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} = 0$ .
- (iv) If  $h$  is affine outside of a compact subset of  $\mathbb{X}$ , then there is a compact subset of  $\mathbb{X}$  outside which  $h_\ell \equiv h$ , for all  $\ell = 1, 2, \dots$

We conclude this introduction with a sketch of the proof. The construction of an approximating diffeomorphism involves five consecutive modifications of  $h$ . Steps 1, 2, and 4 are  $p$ -harmonic replacements based on the Alessandrini-Sigalotti extension [3] of the Radó-Kneser-Choquet Theorem. The other steps involve an explicit smoothing procedure along crosscuts. For this, we adopted some lines of arguments used in J. Munkres' work [32].

## 2. $p$ -HARMONIC MAPPINGS AND PRELIMINARIES

Let  $\Omega$  be a bounded domain in the complex plane  $\mathbb{C} \simeq \mathbb{R}^2$ . A function  $u: \Omega \rightarrow \mathbb{R}$  in the Sobolev class  $\mathcal{W}_{\text{loc}}^{1,p}(\Omega)$ ,  $1 < p < \infty$ , is called  $p$ -harmonic if

$$(2.1) \quad \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

meaning that

$$(2.2) \quad \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle = 0 \quad \text{for every } \varphi \in \mathcal{C}_0^\infty(\Omega).$$

The first observation is that the gradient map  $f = \nabla u: \Omega \rightarrow \mathbb{R}^2$  is  $K$ -quasiregular with  $1 \leq K \leq \max\{p-1, 1/(p-1)\}$ , see [12]. Consequently  $u \in \mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$  with some  $0 < \alpha = \alpha(p) \leq 1$ . In fact [25] the foremost regularity of a  $p$ -harmonic function ( $p \neq 2$ ) is  $\mathcal{C}_{\text{loc}}^{k,\alpha}(\Omega)$ , where the integer  $k \geq 1$  and the Hölder exponent  $\alpha \in (0, 1]$  are determined by the equation

$$k + \alpha = \frac{7p - 6 + \sqrt{p^2 + 12p - 12}}{6p - 6} > 1 + \frac{1}{3}.$$

Thus, regardless of the exponent  $p$ , we have  $u \in \mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$  with  $\alpha = 1/3$ . Clearly, by elliptic regularity theory, outside the singular set

$$\mathcal{S} = \{z \in \Omega: \nabla u(z) = 0\},$$

we have  $u \in \mathcal{C}^\infty(\Omega \setminus \mathcal{S})$ . The singular set, being the set of zeros of a quasiregular mapping, consists of isolated points; unless  $u \equiv \text{const}$ . Pertaining to regularity up to the boundary, we consider a domain  $\Omega$  whose boundary near a point  $z_0 \in \partial\Omega$  is a  $\mathcal{C}^\infty$ -smooth arc, say  $\Gamma \subset \partial\Omega$ . Precisely, we assume that there exist a disk  $D = D(z_0, \epsilon)$  and a  $\mathcal{C}^\infty$ -smooth diffeomorphism  $\varphi: D \xrightarrow{\text{ontq}} \mathbb{C}$  such that

$$\varphi(D \cap \Omega) = \mathbb{C}_+ = \{z: \operatorname{Im} z > 0\}$$

$$\varphi(\Gamma) = \mathbb{R} = \{z: \operatorname{Im} z = 0\}$$

$$\varphi(D \setminus \overline{\Omega}) = \mathbb{C}_- = \{z: \operatorname{Im} z < 0\}.$$

**Proposition 2.1** (Boundary Regularity). *Suppose  $u \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$  is  $p$ -harmonic in  $\Omega$  and  $\mathcal{C}^\infty$ -smooth when restricted to  $\Gamma$ . Then  $u$  is  $\mathcal{C}^{1,\alpha}$ -regular up to  $\Gamma$ , meaning that  $u$  extends to  $D$  as a  $\mathcal{C}^{1,\alpha}(D)$ -regular function, where  $\alpha$  depends only on  $p$ .*

**2.1. The Dirichlet problem.** There are two formulations of the Dirichlet boundary value problem for  $p$ -harmonic equation; both are essential for our investigation. We begin with the variational formulation.

**Lemma 2.2.** *Let  $u_\circ \in \mathcal{W}^{1,p}(\Omega)$  be a given Dirichlet data. There exists precisely one function  $u \in u_\circ + \mathcal{W}_\circ^{1,p}(\Omega)$  which minimizes the  $p$ -harmonic energy:*

$$\mathcal{E}_p[u] = \inf \left\{ \int_\Omega |\nabla w|^p : w \in u_\circ + \mathcal{W}_\circ^{1,p}(\Omega) \right\}.$$

The solution  $u$  is certainly a  $p$ -harmonic function, so  $\mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$ -regular. However, more efficient to us will be the following classical formulation of the Dirichlet problem.

**Problem 2.3.** Given  $u_\circ \in \mathcal{C}(\partial\Omega)$  find a  $p$ -harmonic function  $u$  in  $\Omega$  which extends continuously to  $\overline{\Omega}$  such that  $u|_{\partial\Omega} = u_\circ$ .

It is not difficult to see that such solution (if exists) is unique. However, the existence poses rather delicate conditions on  $\partial\Omega$  and the data  $u_\circ \in \mathcal{C}(\overline{\Omega})$ . We shall confine ourselves to Jordan domains  $\Omega \subset \mathbb{C}$  and the Dirichlet data  $u_\circ \in \mathcal{C}(\overline{\Omega})$  of finite  $p$ -harmonic energy. In this case both formulations are valid and lead to the same solution. Indeed, the variational solution is continuous up to the boundary because each boundary point of a planar Jordan domain is a regular point for the  $p$ -Laplace operator  $\Delta_p$  [18, p.418]. See [22, 6.16] for the discussion of boundary regularity and relevant capacities and [27, Lemma 2] for a capacity estimate that applies to simply connected domains.

**Proposition 2.4** (Existence). *Let  $\Omega \subset \mathbb{C}$  be a bounded Jordan domain and  $u_\circ \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ . There exists, unique,  $p$ -harmonic function  $u \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$  such that  $u|_{\partial\Omega} = u_\circ|_{\partial\Omega}$ .*

**2.2. Radó-Kneser-Choquet Theorem.** Let  $h = u + iv$  be a complex harmonic mapping in a Jordan domain  $\mathbb{U}$  that is continuous on  $\overline{\mathbb{U}}$ . Assume that the boundary mapping  $h : \partial\mathbb{U} \xrightarrow{\text{onto}} \Gamma$  is an orientation-preserving homeomorphism onto a convex Jordan curve. Then  $h$  is a  $\mathcal{C}^\infty$ -smooth diffeomorphism of  $\mathbb{U}$  onto the bounded component of  $\mathbb{C} \setminus \Gamma$ . Thus, in particular, the Jacobian determinant  $J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2$  is strictly positive in  $\mathbb{U}$ , see [15, p.20]. Suppose, in addition, that  $\partial\mathbb{U}$  contains a  $\mathcal{C}^\infty$ -smooth arc  $\gamma \subset \partial\mathbb{U}$ , and  $h$  takes  $\gamma$  onto a  $\mathcal{C}^\infty$ -smooth subarc in  $\Gamma$ . Then  $h$  is  $\mathcal{C}^\infty$ -smooth up to  $\gamma$  and its Jacobian determinant is positive on  $\gamma$  as well, see [15, p.116]. Numerous presentations of the proof of Radó-Kneser-Choquet Theorem can be found, [15]. The idea that goes back to Kneser [26] and Choquet [13]

is to look at the structure of the level curves of the coordinate functions  $u = \operatorname{Re} h$ ,  $v = \operatorname{Im} h$  and their linear combinations. These ideas have been applied to more general linear and nonlinear elliptic systems of PDEs in the complex plane [8], see also [1, 2, 28, 29] for related problems concerning critical points. In the present paper we shall explore a result due to G. Alessandrini and M. Sigalotti [3] for a nonlinear system that consists of two  $p$ -harmonic equations

$$\begin{cases} \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \\ \operatorname{div} |\nabla v|^{p-2} \nabla v = 0 \end{cases}, \quad 1 < p < \infty, \quad h = u + iv.$$

Call it *uncoupled  $p$ -harmonic system*. The novelty and key element in [3] is the associated single linear elliptic PDE of divergence type (with variable coefficients) for a linear combination of  $u$  and  $v$ . Such combination represents a real part of a quasiregular mapping and, therefore, admits only isolated critical points. We shall not go into their arguments in detail, but instead extract the following  $p$ -harmonic analogue of the Radó-Kneser-Choquet Theorem.

**Theorem 2.5** (G. Alessandrini and M. Sigalotti). *Let  $\mathbb{U}$  be a bounded Jordan domain and  $h = u + iv: \overline{\mathbb{U}} \rightarrow \mathbb{C}$  be a continuous mapping whose coordinate functions  $u, v \in \mathcal{W}^{1,p}(\mathbb{U})$ ,  $1 < p < \infty$ , are  $p$ -harmonic. Suppose that  $h: \partial\mathbb{U} \xrightarrow{\text{onto}} \gamma$  is an orientation-preserving homeomorphism onto a convex Jordan curve  $\gamma$ . Then*

- (i)  *$h$  is a  $\mathcal{C}^\infty$ -diffeomorphism from  $\mathbb{U}$  onto the bounded component of  $\mathbb{C} \setminus \gamma$ . In particular,*

$$J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2 > 0 \quad \text{in } \mathbb{U}.$$

- (ii) *If, in addition,  $\partial\mathbb{U}$  contains a  $\mathcal{C}^\infty$ -smooth arc  $\Gamma \subset \partial\mathbb{U}$  and  $h(\Gamma)$  is a  $\mathcal{C}^\infty$ -smooth subarc in  $\gamma$ , then  $h$  is  $\mathcal{C}^{1,\alpha}$ -regular up to  $\Gamma$ , for some  $0 < \alpha = \alpha(p) < 1$  (actually  $\mathcal{C}^\infty$ ). Moreover  $J(z, h) > 0$  on  $\Gamma$  as well.*

This theorem is a straightforward corollary of Theorem 5.1 in [3]. However, three remarks are in order.

- (1) In their Theorem 5.1 the authors of [3] assume that  $\mathbb{U}$  satisfies an exterior cone condition. This is needed only insofar as to ensure the existence of a continuous extension of a given homeomorphism  $\Phi: \partial\mathbb{U} \rightarrow \gamma$  into  $\mathbb{U}$  whose coordinate functions are  $p$ -harmonic in  $\mathbb{U}$ . Obviously, such an extension is unique, though the  $p$ -harmonic energy need not be finite. Once we have such a mapping the exterior cone condition on  $\mathbb{U}$  for the conclusion of Theorem 5.1 is redundant, see Remark 3.2 in [3]. This is exactly the case we are dealing with in Theorem 2.5.
- (2) In regard to the statement (ii) we point out that in Theorem 5.1 of [3] the authors work with the mappings that are smooth up to the entire boundary of  $\mathbb{U}$ . Nonetheless their proof that  $J(z, h) > 0$  on  $\partial\mathbb{U}$  is local, so applies without any change to our case (ii).

- (3) Since  $J(z, h) > 0$  in  $\mathbb{U}$  up to the arc  $\Gamma \subset \partial\mathbb{U}$  the coordinate functions of  $h$  have nonvanishing gradient. This means that  $p$ -harmonic equation is uniformly elliptic up to  $\Gamma$ . Consequently,  $h$  is  $\mathcal{C}^\infty$ -smooth on  $\mathbb{U}$  up to  $\Gamma$ .

**2.3. The  $p$ -harmonic replacement.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2 \simeq \mathbb{C}$ . We consider a class  $\mathcal{A}(\Omega) = \mathcal{A}^p(\Omega)$ ,  $1 < p < \infty$ , of uniformly continuous functions  $h = u + iv: \Omega \rightarrow \mathbb{C}$  having finite  $p$ -harmonic energy and furnish it with the norm

$$\|h\|_{\mathcal{A}^p(\Omega)} = \|h\|_{\mathcal{C}(\Omega)} + \|\nabla h\|_{\mathcal{L}^p(\Omega)}.$$

The closure of  $\mathcal{C}_0^\infty(\Omega)$  in  $\mathcal{A}^p(\Omega)$  will be denoted by  $\mathcal{A}_0^p(\Omega)$ .

**Proposition 2.6.** *Let  $\mathbb{U} \Subset \Omega$  be a Jordan subdomain of  $\Omega$ . There exists a unique operator*

$$\begin{aligned} \mathbf{R}_{\mathbb{U}}: \mathcal{A}^p(\Omega) &\rightarrow \mathcal{A}^p(\Omega) \\ (\text{nonlinear if } p \neq 2) \text{ such that for every } h \in \mathcal{A}^p(\Omega) \\ \mathbf{R}_{\mathbb{U}}h &= h \quad \text{in } \Omega \setminus \mathbb{U} \\ \mathbf{R}_{\mathbb{U}} &\in h + \mathcal{W}_0^{1,p}(\mathbb{U}) \\ \Delta_p \mathbf{R}_{\mathbb{U}}h &= 0 \quad \text{in } \mathbb{U} \end{aligned} \tag{2.3}$$

$$\mathbf{E}_\Omega[\mathbf{R}_{\mathbb{U}}h] \leq \mathbf{E}_\Omega[h] \tag{2.4}$$

Equality occurs in (2.4) if and only if  $h$  is  $p$ -harmonic in  $\mathbb{U}$ .

*Proof.* For  $h = u + iv$  we define

$$\mathbf{R}_{\mathbb{U}}h = \mathbf{R}_{\mathbb{U}}u + i \mathbf{R}_{\mathbb{U}}v.$$

It is therefore enough to construct the replacement for real-valued functions. For  $u \in \mathcal{A}^p(\Omega)$  real, we define

$$\mathbf{R}_{\mathbb{U}}u = \begin{cases} u & \text{in } \Omega \setminus \mathbb{U} \\ \tilde{u} & \text{in } \mathbb{U} \end{cases}$$

where  $\tilde{u}$  is determined uniquely as a solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} = 0 & \text{in } \mathbb{U} \\ \tilde{u} \in u + \mathcal{W}_0^{1,p}(\mathbb{U}) \end{cases}$$

so conditions (2.3) are fulfilled. That  $\mathbf{R}_{\mathbb{U}}u$  is continuous in  $\Omega$  is guaranteed by Proposition 2.4. The solution  $\tilde{u}$  is found as the minimizer of the  $p$ -harmonic energy in the class  $u + \mathcal{W}_0^{1,p}(\mathbb{U})$ , so we certainly have

$$\mathbf{E}_\Omega[\mathbf{R}_{\mathbb{U}}u] \leq \mathbf{E}_\Omega[u]$$

The same estimate holds for the imaginary part of  $h$ , so adding them up yields

$$\mathbf{E}_\Omega[\mathbf{R}_{\mathbb{U}}h] \leq \mathbf{E}_\Omega[h]. \quad \square$$

*Remark 2.7.* The reader may wish to know that the operator  $\mathbf{R}_{\mathbb{U}}: \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$  is continuous, though we do not appeal to this fact.

**2.4. Smoothing along a crosscut.** Consider a bounded Jordan domain  $\mathbb{U}$  and a  $\mathcal{C}^\infty$ -smooth crosscut  $\Gamma \subset \mathbb{U}$  with two distinct end-points in  $\partial\mathbb{U}$ . By definition, this means that there is a  $\mathcal{C}^\infty$ -diffeomorphism  $\varphi: \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$  such that  $\Gamma = \varphi(\mathbb{R})$ , and its distinct endpoints are given by

$$\begin{aligned} \lim_{x \rightarrow -\infty} \varphi(x) &\in \partial\mathbb{U} \\ \lim_{x \rightarrow \infty} \varphi(x) &\in \partial\mathbb{U} \end{aligned}$$

Such  $\Gamma$  splits  $\mathbb{U}$  into two Jordan subdomains

$$\begin{aligned} \mathbb{U}_+ &= \varphi(\mathbb{C}_+), \quad \mathbb{C}_+ = \{z: \text{Im } z > 0\} \\ \mathbb{U}_- &= \varphi(\mathbb{C}_-), \quad \mathbb{C}_- = \{z: \text{Im } z < 0\}. \end{aligned}$$

Suppose we are given a homeomorphism  $f: \overline{\mathbb{U}} \rightarrow \mathbb{C}$  such that each of two mappings

$$f: \mathbb{U}_+ \rightarrow \mathbb{R}^2 \quad \text{and} \quad f: \mathbb{U}_- \rightarrow \mathbb{R}^2$$

is  $\mathcal{C}^\infty$ -smooth up to  $\Gamma$ . Assume that for some constant  $0 < m < \infty$  we have

$$|Df(z)| \leq m \quad \text{and} \quad \det Df(z) \geq \frac{1}{m}$$

on  $\mathbb{U}_+$  and on  $\mathbb{U}_-$ . Thus  $f: \mathbb{U} \rightarrow \mathbb{R}^2$  is in fact locally bi-Lipschitz.

**Proposition 2.8.** *Under the above conditions there is a constant  $0 < M < \infty$  such that for every open set  $\mathbb{V} \subset \mathbb{U}$  containing  $\Gamma$  one can find a homeomorphism  $g: \overline{\mathbb{U}} \xrightarrow{\text{onto}} f(\overline{\mathbb{U}})$  which is a  $\mathcal{C}^\infty$ -diffeomorphism in  $\mathbb{U}$ , with the following properties:*

$$(2.5) \quad g(z) = f(z), \text{ for } z \in (\overline{\mathbb{U}} \setminus \mathbb{V}) \cup \Gamma$$

$$(2.6) \quad |Dg(z)| \leq M \quad \text{and} \quad \det Dg(z) > \frac{1}{M} \text{ on } \mathbb{U}.$$

The key element of this smoothing device is that the constant  $M$  is independent of the neighborhood  $\mathbb{V}$  of  $\Gamma$ , see Figure 1. The proof is given in [24] following the ideas of [32].

We shall recall similar smoothing device for cuts along Jordan curves. Let  $\mathbb{U}$  be a simply connected domain with  $\mathcal{C}^\infty$ -regular cut along a Jordan curve  $\Gamma \subset \mathbb{U}$ . This means there is a diffeomorphism  $\varphi: \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$  such that  $\Gamma = \varphi(\mathbb{S}^1)$ ,  $\mathbb{S}^1 = \{z \in \mathbb{C}: |z| = 1\}$ . As before  $\Gamma$  splits  $\mathbb{U}$  into

$$\begin{aligned} \mathbb{U}_+ &= \varphi(\mathbb{D}_+), \quad \mathbb{D}_+ = \{z: |z| < 1\} \\ \mathbb{U}_- &= \varphi(\mathbb{D}_-), \quad \mathbb{D}_- = \{z: |z| > 1\}. \end{aligned}$$

Suppose we are given a homeomorphism  $f: \mathbb{U} \rightarrow \mathbb{R}^2$  such that each of two mappings

$$f: \mathbb{U}_+ \rightarrow \mathbb{R}^2 \quad \text{and} \quad f: \mathbb{U}_- \rightarrow \mathbb{R}^2$$

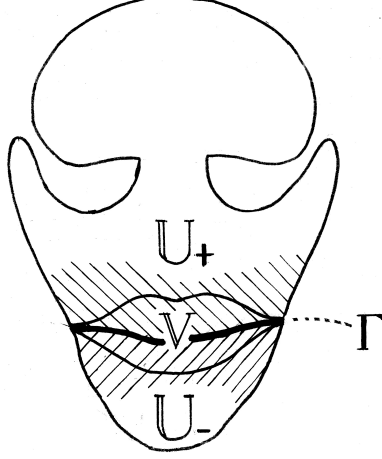


FIGURE 1. Jordan domain with a crosscut  $\Gamma$  and its neighborhood  $\mathbb{V}$ .

is  $\mathcal{C}^\infty$ -smooth up to  $\Gamma$ . Assume that for some constant  $0 < m < \infty$  we have

$$|Df(z)| \leq m \quad \text{and} \quad \det Df(z) \geq \frac{1}{m}$$

on  $\mathbb{U}_+$  and  $\mathbb{U}_-$ .

**Proposition 2.9.** *Under the above conditions there is a constant  $0 < M < \infty$  such that for every open set  $\mathbb{V} \subset \mathbb{U}$  containing  $\Gamma$  one can find a  $\mathcal{C}^\infty$ -diffeomorphism  $g: \mathbb{U} \xrightarrow{\text{onto}} f(\mathbb{U})$  with the following properties*

$$(2.7) \quad g(z) = f(z), \text{ for } z \in (\mathbb{U} \setminus \mathbb{V}) \cup \Gamma$$

$$(2.8) \quad |Dg(z)| \leq M \quad \text{and} \quad \det Dg(z) > \frac{1}{M} \text{ on } \mathbb{U}.$$

Having disposed of the above preliminaries we shall now proceed to the construction of the approximating sequence of diffeomorphisms.

### 3. THE PROOF

**3.1. Scheme of the proof.** Let us begin with a convention. We will often suppress the explicit dependence on the Sobolev exponent  $1 < p < \infty$  in the notation, whenever it becomes selfexplanatory. For every  $\epsilon > 0$  we shall construct a  $\mathcal{C}^\infty$ -diffeomorphism  $\tilde{h}: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$  such that

- (A)  $\tilde{h} - h \in \mathcal{A}_0(\mathbb{X})$
- (B)  $\|\tilde{h} - h\|_{\mathcal{C}(\mathbb{X})} \leq \epsilon$
- (C)  $\|\nabla \tilde{h} - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} \leq \epsilon$
- (D)  $E_{\mathbb{X}}[\tilde{h}] \leq E_{\mathbb{X}}[h]$



- (E) If  $h$  is a  $\mathcal{C}^\infty$ -diffeomorphism outside of a compact subset of  $\mathbb{X}$ , then there exist a compact subset of  $\mathbb{X}$  outside of which we have  $\tilde{h} \equiv h$ , for all  $\epsilon > 0$ .

We may and do assume that  $h$  is not a  $\mathcal{C}^\infty$ -diffeomorphism, since otherwise  $\tilde{h} = h$  satisfies the desired properties. Let  $x_o \in \mathbb{X}$  be a point such that  $h$  fails to be  $\mathcal{C}^\infty$ -diffeomorphism in any neighborhood of  $x_o$ .

We shall consider dyadic squares in  $\mathbb{Y}$  with respect to a selected rectangular coordinate system in  $\mathbb{R}^2$ . By choosing the origin of the system we ensure that  $h(x_o)$  does not lie on the boundary of any dyadic square.

Let us fix  $\epsilon > 0$ . The construction of  $\tilde{h}$  proceeds in 5 steps, each of which gives a homeomorphism  $\tilde{h}_k: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ,  $k = 0, 1, \dots, 5$ , in the Sobolev class  $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$  such that  $\tilde{h}_0 = h$ ,  $\tilde{h}_k \in \tilde{h}_{k-1} + \mathcal{A}_o(\mathbb{X})$ ,  $k = 1, \dots, 5$  and  $\tilde{h}_5 = \tilde{h}$  is the desired diffeomorphism. For each  $k = 1, 2, \dots, 5$  we will secure conditions analogous to (A)-(E). Namely,

- (A<sub>k</sub>)  $\tilde{h}_k - \tilde{h}_{k-1} \in \mathcal{A}_o(\mathbb{X})$
- (B<sub>k</sub>)  $\|\tilde{h}_k - \tilde{h}_{k-1}\|_{\mathcal{C}(\mathbb{X})} \leq \epsilon/5$
- (C<sub>k</sub>)  $\|\nabla \tilde{h}_k - \nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})} \leq \epsilon/5$
- (D<sub>k</sub>)  $\|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_0\|_{\mathcal{L}^p(\mathbb{X})} - 2\delta$ , for some  $\delta > 0$ ;  
 $\|\nabla \tilde{h}_k\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})}$ , for  $k = 2, 4$ ;  
 $\|\nabla \tilde{h}_k\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})} + \delta$ , for  $k = 3, 5$
- (E<sub>k</sub>) If  $\tilde{h}_{k-1}$  is a  $\mathcal{C}^\infty$ -diffeomorphism outside of a compact subset of  $\mathbb{X}$ , then there exists a compact subset in  $\mathbb{X}$  outside which we have  $\tilde{h}_k \equiv \tilde{h}_{k-1}$  for all  $\epsilon > 0$ .

**3.2. Partition of  $\mathbb{X}$  into cells.** Let us distinguish one particular Whitney type partition of  $\mathbb{Y}$  and keep it fixed for the rest of our arguments.

$$\mathbb{Y} = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{Y}_\nu},$$

where  $\mathbb{Y}_\nu$  are mutually disjoint open dyadic squares such that

$$\text{diam } \mathbb{Y}_\nu \leq \text{dist}(\mathbb{Y}_\nu, \partial \mathbb{Y}) \leq 3 \text{diam } \mathbb{Y}_\nu \quad \text{for } \nu = 1, 2, \dots$$

unless  $\mathbb{Y} = \mathbb{R}^2$ , in which case  $\mathbb{Y}_\nu$  are unit squares. Thus the cover of  $\mathbb{Y}$  by  $\overline{\mathbb{Y}_\nu}$  is locally finite. The preimages

$$\mathbb{X}_\nu = h^{-1}(\mathbb{Y}_\nu), \quad \nu = 1, 2, \dots$$

are Jordan domains which we call *cells* in  $\mathbb{X}$ . In the forthcoming Step 1 we shall need to further divide each cell into a finite number of *daughter cells* in  $\mathbb{X}$ . Note that all but finite number of cells  $\mathbb{X}_\nu$ ,  $\nu = 1, 2, \dots$  lie outside a given compact subset of  $\mathbb{X}$ .

#### STEP 1

To avoid undue indexing in the forthcoming division of cells, we shall argue in two substeps.

**Step 1a.** Examine one of the cells in  $\mathbb{X}$ , say  $\mathfrak{X} = \mathbb{X}_\nu$ , for some fixed  $\nu = 1, 2, \dots$ . Call it a *parent cell*. Thus  $h(\mathfrak{X}) = \Upsilon$  is the corresponding Whitney square  $\Upsilon = \mathbb{Y}_\nu \subset \mathbb{Y}$ . To every  $n = 1, 2, \dots$ , there corresponds a partition of  $\Upsilon$  into  $4^n$ -dyadic congruent squares  $\Upsilon_i$ ,  $i = 1, \dots, 4^n$

$$\overline{\Upsilon} = \overline{\Upsilon_1} \cup \dots \cup \overline{\Upsilon_{4^n}}.$$

This gives rise to a division of  $\mathfrak{X}$  into daughter cells  $\mathfrak{X}_i = h^{-1}(\Upsilon_i)$

$$\overline{\mathfrak{X}} = \overline{\mathfrak{X}_1} \cup \overline{\mathfrak{X}_2} \cup \dots \cup \overline{\mathfrak{X}_{4^n}}.$$

We look at the homeomorphisms

$$h: \overline{\mathfrak{X}_i} \xrightarrow{\text{ontq}} \overline{\Upsilon_i}, \quad i = 1, 2, \dots, 4^n$$

By virtue of Proposition 2.6 we may replace them with  $p$ -harmonic homeomorphisms

$$\tilde{h}_i = \mathbf{R}_{\mathfrak{X}_i} h: \overline{\mathfrak{X}_i} \xrightarrow{\text{ontq}} \overline{\Upsilon_i}, \quad i = 1, 2, \dots, 4^n$$

which coincide with  $h$  on  $\partial\mathfrak{X}_i$ . This procedure may not be necessary if  $h: \mathfrak{X}_i \rightarrow \Upsilon_i$  is already a  $\mathcal{C}^\infty$ -diffeomorphism. In such cases we always use the *trivial replacement*  $\tilde{h}_i = h$ . After all such replacements are made, we arrive at a homeomorphism

$$\tilde{h}: \overline{\mathfrak{X}} \xrightarrow{\text{ontq}} \overline{\Upsilon}$$

which is a  $\mathcal{C}^\infty$ -diffeomorphism in each cell  $\mathfrak{X}_i$  and coincides with  $h$  on  $\partial\mathfrak{X}_i$ . Obviously,

$$\tilde{h} = h + \sum_{i=1}^{4^n} [\tilde{h}_i - h]_\circ \in h + \mathcal{A}_\circ(\mathfrak{X})$$

where  $[\tilde{h}_i - h]_\circ$  stands for zero extension of  $\tilde{h}_i - h$  outside  $\mathfrak{X}_i$  and, therefore, belongs to  $\mathcal{A}_\circ(\mathfrak{X}_i)$ . Furthermore, by principle of minimal  $p$ -harmonic energy, we have

$$\mathbf{E}_{\mathfrak{X}}[\tilde{h}] = \sum_{i=1}^{4^n} \mathbf{E}_{\mathfrak{X}_i}[\tilde{h}_i] \leq \sum_{i=1}^{4^n} \mathbf{E}_{\mathfrak{X}_i}[h] = \mathbf{E}_{\mathfrak{X}}[h].$$

The eventual aim is to fix the number of daughter cells in  $\mathfrak{X}$ . For this we vary  $n$  and look closely at the resulting homeomorphisms, denoted by  $f_n$ . This sequence of mappings is bounded in  $\mathcal{A}(\mathfrak{X})$ . It actually converges to  $h$  uniformly on  $\overline{\mathfrak{X}}$ . Indeed, given any point  $x \in \overline{\mathfrak{X}}$ , say  $x \in \overline{\mathfrak{X}_i}$ , for some  $i = 1, 2, \dots, 4^n$ , we have

$$|f_n(x) - h(x)| = |\tilde{h}_i(x) - h(x)| \leq \text{diam } \Upsilon_i = 2^{-n} \text{diam } \Upsilon.$$

Thus

$$\lim_{n \rightarrow \infty} f_n = h, \quad \text{uniformly in } \overline{\mathfrak{X}}.$$

On the other hand the mappings  $f_n$  are bounded in the Sobolev space  $\mathcal{W}^{1,p}(\mathfrak{X})$ , so converge to  $h$  weakly in  $\mathcal{W}^{1,p}(\mathfrak{X})$ . The key observation now is that

$$\|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})} \leq \liminf_{n \rightarrow \infty} \|\nabla f_n\|_{\mathcal{L}^p(\mathfrak{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})}$$

because of convexity of the energy functional. This gives

$$\lim_{n \rightarrow \infty} \|\nabla f_n\|_{\mathcal{L}^p(\mathfrak{X})} = \|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})}$$

Then, the usual application of Clarkson's inequalities in  $\mathcal{L}^p$ -spaces,  $1 < p < \infty$ , yields

$$\lim_{n \rightarrow \infty} \|\nabla f_n - \nabla h\|_{\mathcal{L}^p(\mathfrak{X})} = 0$$

meaning that  $f_n - h \rightarrow 0$  in the norm topology of  $\mathcal{A}(\mathfrak{X})$ . We can now determine the number  $n = n_\nu = n(\mathfrak{X})$ , simply requiring the division of  $\mathfrak{X}$  be fine enough to satisfy two conditions.

$$(3.1) \quad \begin{cases} \text{diam } \Upsilon_i = 2^{-n} \text{diam } \Upsilon \leq \epsilon/5, & i = 1, \dots, 4^n \\ \|\nabla f_n - \nabla h\|_{\mathcal{L}^p(\mathfrak{X})} \leq \frac{\epsilon}{5 \cdot 2^\nu} \end{cases}$$

where we recall that  $\mathfrak{X}$  stands for  $\mathbb{X}_\nu$ .

**Step 1b.** Now, having  $n = n_\nu$  fixed for each cell  $\mathfrak{X}_\nu$ , we construct our first approximating mapping

$$\tilde{h}_1: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

by setting

$$\tilde{h}_1 := h + \sum_{\nu=1}^{\infty} [f_{n_\nu} - h]_\circ \in h + \mathcal{A}_\circ(\mathbb{X})$$

where, as always,  $[f_{n_\nu} - h]_\circ$  stands for the zero extension of  $f_{n_\nu} - h$  outside  $\mathbb{X}_\nu$ . This mapping is a  $\mathcal{C}^\infty$ -diffeomorphism in every daughter cell. Clearly, we have the condition

$$(A_1) \quad \tilde{h}_1 - h \in \mathcal{A}_\circ(\mathbb{X}).$$

Moreover, by the condition in (3.1) imposed on every  $n_\nu$ ,

$$(B_1) \quad \|\tilde{h}_1 - h\|_{\mathcal{C}(\mathbb{X})} \leq \sup_{\nu=1,2,\dots} \{\text{diam } \Upsilon_i: \Upsilon_i \subset \mathbb{Y}_\nu, i = 1, \dots, 4^{n_\nu}\} \leq \frac{\epsilon}{5}$$

and

$$(C_1) \quad \|\nabla \tilde{h}_1 - \nabla h\|_{\mathcal{L}^p(\mathbb{X})}^p = \sum_{\nu=1}^{\infty} \|\nabla \tilde{h}_1 - \nabla h\|_{\mathcal{L}^p(\mathbb{X}_\nu)}^p \leq \left(\frac{\epsilon}{5}\right)^p \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} < \left(\frac{\epsilon}{5}\right)^p.$$

Regarding condition  $(D_1)$ , we observe that summing up the energies over all daughter cells  $\mathfrak{X}_i \subset \mathbb{X}_\nu$ ,  $i = 1, 2, \dots, 4^{n_\nu}$  and  $\nu = 1, 2, \dots$ , gives the total energy of  $\tilde{h}_1$  not larger than that of  $h$ . Even more, since  $h$  fails to be a  $\mathcal{C}^\infty$ -diffeomorphism in at least one of these cells, the  $p$ -harmonic replacement takes place in this cell and, consequently,  $\tilde{h}_1$  has strictly smaller energy. Hence

$$(D_1) \quad \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})} - 2\delta, \quad \text{for some } \delta > 0.$$

Regarding condition  $(E_1)$ , we note that under the assumption therein we made only a finite number of nontrivial ( $p$ -harmonic) replacements. The same remark will apply to the subsequent steps and will not be mentioned again. The step 1 is complete.

Before proceeding to Step 2, let us put all daughter cells in  $\mathbb{X}$  in a single sequence

$$\mathfrak{X}^1, \mathfrak{X}^2, \dots \subset \mathbb{X}.$$

Thus from now on the daughter cells from different parents are indistinguishable as far as the mapping  $h_1$  is concerned. The point is that  $h_1$  is a  $\mathcal{C}^\infty$ -diffeomorphism in every such cell, a property that will be pertinent to all new cells coming later either by splitting or merging the existing cells. Note that the images  $\Upsilon^\alpha = h(\mathfrak{X}^\alpha)$ ,  $\alpha = 1, 2, \dots$ , form a partition of  $\mathbb{Y}$  into dyadic squares

$$\mathbb{Y} = \bigcup_{\alpha=1}^{\infty} \overline{\Upsilon^\alpha}, \quad \text{where} \quad \text{diam } \Upsilon^\alpha \leq \frac{\epsilon}{5}.$$

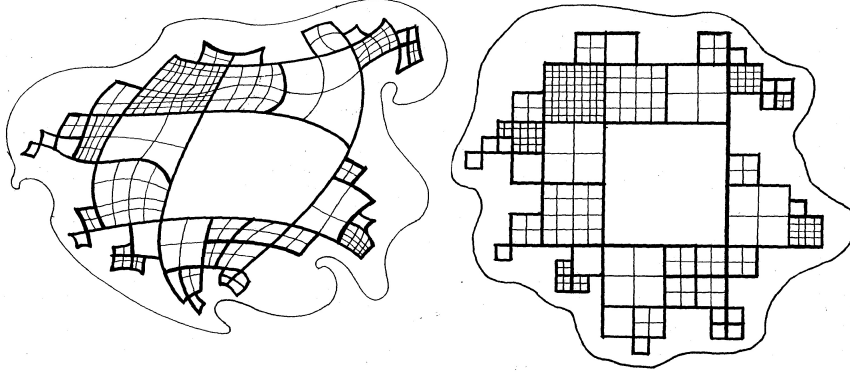


FIGURE 2.  $h_1$  is a  $\mathcal{C}^\infty$ -diffeomorphism in each cell  $\mathfrak{X}^\alpha \subset \mathbb{X}$ .

## STEP 2

**Step 2a.** (Adjacent cells) Let  $\mathcal{C}(\mathbb{Y}) \subset \mathbb{Y}$  be the collection of all corners of dyadic squares  $\Upsilon^\alpha$ ,  $\alpha = 1, 2, \dots$ , and  $\mathcal{V}(\mathbb{X}) \subset \mathbb{X}$  denote the set of their preimages under  $h$ , called *vertices of cells*. Whenever two closed cells  $\overline{\mathfrak{X}^\alpha}$  and  $\overline{\mathfrak{X}^\beta}$ ,  $\alpha \neq \beta$ , intersect, their common part is either a point in  $\mathcal{V}(\mathbb{X})$  or an edge, that is, a closed Jordan arc with endpoints in  $\mathcal{V}(\mathbb{X})$ . In this latter case we say that  $\mathfrak{X}^\alpha$  and  $\mathfrak{X}^\beta$  are adjacent cells with common edge

$$\overline{C^{\alpha\beta}} = \overline{\mathfrak{X}^\alpha} \cap \overline{\mathfrak{X}^\beta}.$$

This is the closure of a Jordan open arc  $C^{\alpha\beta} = \overline{C^{\alpha\beta}} \setminus \mathcal{V}(\mathbb{X})$ . The mappings

$$h_1: \mathfrak{X}^\alpha \xrightarrow{\text{onto}} \Upsilon^\alpha \quad \text{and} \quad h_1: \mathfrak{X}^\beta \xrightarrow{\text{onto}} \Upsilon^\beta$$

are  $\mathcal{C}^\infty$ -diffeomorphisms but they do not necessarily match smoothly along the edges. We shall now produce a new cell  $\mathfrak{X}^{\alpha\beta}$ , a daughter of the adjacent cells  $\mathfrak{X}^\alpha$  and  $\mathfrak{X}^\beta$ , such that

$$C^{\alpha\beta} \subset \mathfrak{X}^{\alpha\beta} \subset \mathfrak{X}^\alpha \cup C^{\alpha\beta} \cup \mathfrak{X}^\beta.$$

To construct  $\mathfrak{X}^{\alpha\beta}$  we look at the adjacent dyadic squares  $\overline{\Upsilon^\alpha}$  and  $\overline{\Upsilon^\beta}$  in  $\mathbb{Y}$ . The intersection  $\overline{\Upsilon^\alpha} \cap \overline{\Upsilon^\beta} = h(\overline{C^{\alpha\beta}})$  is a closed interval. Let  $R$  be a number greater than the length of  $h(C^{\alpha\beta})$  to be chosen sufficiently large later on. There exist exactly two open disks of radius  $R$  for which  $h(C^{\alpha\beta})$  is a chord. Their intersection, denoted by  $\mathcal{L}^{\alpha\beta}$ , is a symmetric doubly convex lens of curvature  $R^{-1}$ . Thus  $\mathcal{L}^{\alpha\beta}$  is enclosed between two open circular arcs  $\gamma^{\alpha\beta} = \Upsilon^\alpha \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\alpha$  and  $\gamma^{\beta\alpha} = \Upsilon^\beta \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\beta$ . Note that  $\mathcal{L}^{\alpha\beta} = \mathcal{L}^{\beta\alpha}$ , but  $\gamma^{\alpha\beta} \neq \gamma^{\beta\alpha}$ . We call

$$(3.2) \quad \mathfrak{X}^{\alpha\beta} = h_1^{-1}(\mathcal{L}^{\alpha\beta}), \quad \text{a daughter of the adjacent cells } \mathfrak{X}^\alpha \text{ and } \mathfrak{X}^\beta.$$

As the curvature of the lens  $\mathcal{L}^{\alpha\beta}$  approaches zero, the area of  $\mathfrak{X}^{\alpha\beta}$  tends to 0. This allows us to choose  $R$  so that

$$(3.3) \quad \|\nabla h_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \leq \frac{\epsilon}{5 \cdot 2^{\alpha+\beta}}.$$

The lenses  $\mathcal{L}^{\alpha\beta}$  are disjoint because the opening angle of each lens (the angle between arcs at their common endpoints) is at most  $\pi/3$  and their long axes are either parallel or orthogonal, see Figure 3. Therefore, the cells  $\mathfrak{X}^{\alpha\beta} = h_1^{-1}(\mathcal{L}^{\alpha\beta})$  are also disjoint. However, their closures may have a common point that lies in  $\mathcal{V}(\mathbb{X})$ . The boundary of  $\mathfrak{X}^{\alpha\beta}$  consists of two open arcs

$$\Gamma^{\alpha\beta} = \mathfrak{X}^\alpha \cap \partial\mathfrak{X}^{\alpha\beta} \quad \text{and} \quad \Gamma^{\beta\alpha} = \mathfrak{X}^\beta \cap \partial\mathfrak{X}^{\alpha\beta}$$

plus their endpoints. These open arcs are  $\mathcal{C}^\infty$ -smooth because they come as images of the circular arcs enclosing the lens  $\mathcal{L}^{\alpha\beta}$  under a  $\mathcal{C}^\infty$ -diffeomorphism.

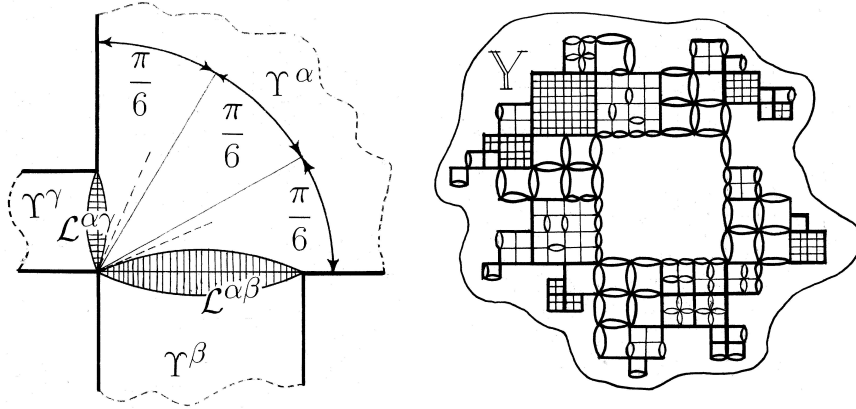


FIGURE 3. Lenses.

*Remark 3.1.* In what follows we shall consider only the pairs  $(\alpha, \beta)$  of indices  $\alpha = 1, 2, \dots$  and  $\beta = 1, 2, \dots$  which correspond to adjacent cells. Such pairs will be designated the symbol  $\alpha\beta$ .

**Step 2b.** (Replacements in  $\mathfrak{X}^{\alpha\beta}$ ) The lenses  $\mathcal{L}^{\alpha\beta} \subset \mathbb{Y}$  are convex, so with the aid of Proposition 2.6 and Theorem 2.5, we may replace  $\tilde{h}_1: \mathfrak{X}^{\alpha\beta} \rightarrow \mathcal{L}^{\alpha\beta}$  with the  $p$ -harmonic extension of  $\tilde{h}_1: \partial\mathfrak{X}^{\alpha\beta} \rightarrow \partial\mathcal{L}^{\alpha\beta}$ . We do this, and denote the result by  $\tilde{h}_2^{\alpha\beta}: \mathfrak{X}^{\alpha\beta} \rightarrow \mathcal{L}^{\alpha\beta}$ , only on the cells in which  $\tilde{h}_1: \mathfrak{X}^\alpha \cup \mathfrak{X}^\beta \cup \mathfrak{X}^{\alpha\beta} \rightarrow \mathbb{R}^2$  is not a  $\mathcal{C}^\infty$ -diffeomorphism. In other cells we set  $\tilde{h}_2^{\alpha\beta} = \tilde{h}_1$ . In either case  $\tilde{h}_2^{\alpha\beta} \in \tilde{h}_1 + \mathcal{A}_o(\mathfrak{X}^{\alpha\beta})$  so we define

$$\tilde{h}_2 = \tilde{h}_1 + \sum_{\alpha\beta} [\tilde{h}_2^{\alpha\beta} - \tilde{h}_1]_o.$$

Thus we have

$$(A_2) \quad \tilde{h}_2 - \tilde{h}_1 \in \mathcal{A}_o(\mathbb{X}).$$

The advantage of using  $\tilde{h}_2$  in the next step lies in the fact that it is not only a  $\mathcal{C}^\infty$ -diffeomorphism in every cell, but also is  $\mathcal{C}^\infty$ -smooth with positive Jacobian determinant, up to each edge of the cells created here. These edges are  $\mathcal{C}^\infty$ -smooth open arcs. By cells created here we mean not only  $\mathfrak{X}^{\alpha\beta}$  but also those obtained from the parent cell  $\mathfrak{X}^\alpha$  by removing the adjacent daughters; that is,

$$\mathfrak{X}^\alpha \setminus \bigcup_{\alpha\beta} \mathfrak{X}^{\alpha\beta}, \quad \alpha = 1, 2, \dots$$

See Figure 4. The estimates of  $\tilde{h}_2$  run as follows. By (3.1) we have,

$$(B_2) \quad \|\tilde{h}_2 - \tilde{h}_1\|_{\mathcal{C}(\mathbb{X})} \leq \sup_{\alpha\beta} \{\text{diam } \mathcal{L}^{\alpha\beta}\} \leq \sup_{\alpha} \{\text{diam } \mathbb{Y}^\alpha\} \leq \frac{\epsilon}{5}.$$

In view of the minimum  $p$ -harmonic energy principle, we have

$$\begin{aligned} \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} &= \sum_{\alpha\beta} \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})} \\ &\leq \sum_{\alpha\beta} \left[ \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \right] \\ &\leq 2 \sum_{\alpha\beta} \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \leq \frac{2\epsilon}{5} \sum_{\alpha\beta} 2^{-\alpha-\beta}. \end{aligned}$$

by (3.3). Hence

$$(C_2) \quad \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

The minimum energy principle also yields estimate

$$\begin{aligned} \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathbb{X})}^p &= \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})}^p + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X} \setminus \cup \mathfrak{X}^{\alpha\beta})}^p \\ &\leq \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})}^p + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X} \setminus \cup \mathfrak{X}^{\alpha\beta})}^p = \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})}^p. \end{aligned}$$

In particular

$$(D_2) \quad \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})},$$

completing the proof of Step 2.

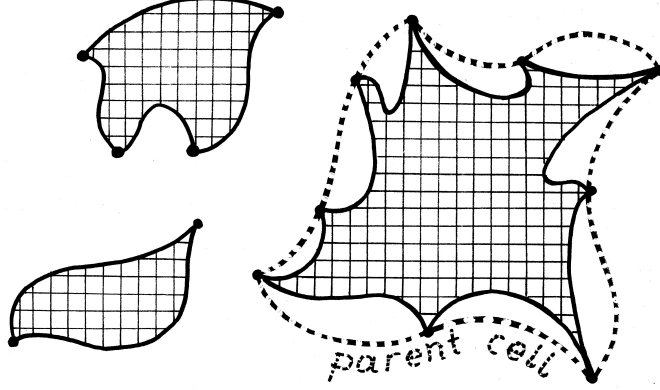


FIGURE 4. Three types of cells.

Note that  $h_2$  is locally bi-Lipschitz in  $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$ . The exceptional set  $\mathcal{V}(\mathbb{X})$  is discrete.

### STEP 3

We shall now merge all the adjacent cells together, by smoothing  $h_2$  around the edges  $\Gamma^{\alpha\beta} \subset \mathfrak{X}^\alpha$ . To achieve proper estimates we need to remove small neighborhoods of all vertices, outside which  $h_2$  is certainly locally bi-Lipschitz.

**Step 3a.** First we cover the set  $\mathcal{C}(\mathbb{Y})$  of corners of dyadic squares by disks  $\mathbb{D}_c$  centered at  $c \in \mathcal{C}(\mathbb{Y})$ . These disks will be chosen small enough to satisfy all the conditions listed below.

(i)  $\text{diam } \mathbb{D}_c < \epsilon/5$  for every  $c \in \mathcal{C}(\mathbb{Y})$ ,

(ii)  $\sum_{v \in \mathcal{V}(\mathbb{X})} \int_{\mathbb{F}_v} |\nabla h_2|^p \leq \left(\frac{\epsilon}{20}\right)^p$ , where  $\mathbb{F}_v = h_2^{-1}(\mathbb{D}_c)$ ,  $c = h_2(v) = h(v)$ .

Denote by  $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup \overline{\mathbb{F}_v}$ . We truncate each edge  $\Gamma^{\alpha\beta}$  near the endpoints by setting

$$(3.4) \quad \Gamma_\circ^{\alpha\beta} = \Gamma^{\alpha\beta} \cap \mathbb{X}_\circ.$$

These are mutually disjoint open arcs; their closures are isolated continua in  $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$ . This means that there are disjoint neighborhoods of them. We are actually interested in neighborhoods  $\mathbb{U}^{\alpha\beta} \subset \mathfrak{X}^\alpha$  of  $\Gamma_\circ^{\alpha\beta}$  that are Jordan domains in which  $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$  are  $\mathcal{C}^\infty$ -smooth crosscuts with two endpoints in  $\partial\mathbb{U}^{\alpha\beta}$ , see Section 2. It is geometrically clear that such mutually disjoint neighborhoods exist. Now the stage for next substep is established.

**Step 3b.** ( $\mathcal{C}^\infty$ -replacement within  $\mathbb{U}^{\alpha\beta}$ ) It is at this stage that we will improve  $h_2$  in  $\mathbb{U}^{\alpha\beta}$  to a  $\mathcal{C}^\infty$ -smooth diffeomorphism with no harm to the previously established estimates for  $h_2$ . The tool is Proposition 2.8. As

always, we shall make no replacement of  $h_2: \mathbb{U}^{\alpha\beta} \rightarrow \Upsilon^\alpha$  if it is already  $\mathcal{C}^\infty$ -diffeomorphism. Recall that we have a bi-Lipschitz mapping  $h_2: \mathbb{U}^{\alpha\beta} \rightarrow h_2(\mathfrak{X}^\alpha) = \Upsilon^\alpha$  that takes the crosscut  $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$  onto a circular arc. Denote the components  $\mathbb{U}_+^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \setminus \overline{\mathfrak{X}^{\alpha\beta}}$  and  $\mathbb{U}_-^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \cap \mathfrak{X}^{\alpha\beta}$ . Furthermore, we have

$$|Dh_2| \leq m_{\alpha\beta} \quad \text{and} \quad \det Dh_2 \geq \frac{1}{m_{\alpha\beta}}, \quad \text{for some } m_{\alpha\beta} > 0$$

on each component. The mappings  $h_2: \mathbb{U}_+^{\alpha\beta} \rightarrow \Upsilon^\alpha$  and  $h_2: \mathbb{U}_-^{\alpha\beta} \rightarrow \Upsilon^\alpha$  are  $\mathcal{C}^\infty$ -diffeomorphisms up to  $\Gamma_\circ^{\alpha\beta}$ . In accordance with Proposition 2.8 we find a constant  $M_{\alpha\beta}$  such that: whenever open set  $\mathbb{V}^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$  contains the crosscut  $\Gamma_\circ^{\alpha\beta}$  there exists a homeomorphism  $h_3^{\alpha\beta}: \overline{\mathbb{U}^{\alpha\beta}} \xrightarrow{\text{ontg}} h_2(\overline{\mathbb{U}^{\alpha\beta}})$  which is a  $\mathcal{C}^\infty$ -diffeomorphism in  $\mathbb{U}^{\alpha\beta}$ , with the following properties

- $h_3^{\alpha\beta} \equiv h_2$  on  $(\overline{\mathbb{U}^{\alpha\beta}} \setminus \mathbb{V}^{\alpha\beta}) \cup \Gamma_\circ^{\alpha\beta}$ ;
- $|\nabla h_3^{\alpha\beta}| \leq M_{\alpha\beta}$  and  $\det \nabla h_3^{\alpha\beta} \geq \frac{1}{M_{\alpha\beta}}$  in  $\mathbb{U}^{\alpha\beta}$ .

Since  $M_{\alpha\beta}$  does not depend on  $\mathbb{V}^{\alpha\beta}$  it will be advantageous to take neighborhoods  $\mathbb{V}^{\alpha\beta}$  of  $\Gamma_\circ^{\alpha\beta}$  thin enough to satisfy

- $\overline{\mathbb{V}^{\alpha\beta}} \subset \mathbb{U}^{\alpha\beta} \cup \overline{\Gamma_\circ^{\alpha\beta}}$ ;
- $|\mathbb{V}^{\alpha\beta}| \leq \frac{1}{5^p \cdot 2^{\alpha+\beta}} \left[ \frac{\epsilon}{m_{\alpha\beta} + M_{\alpha\beta}} \right]^p$  and also  $|\mathbb{V}^{\alpha\beta}| \leq \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}}$ .

Note that  $h_3^{\alpha\beta}, h_2 \in \mathcal{W}^{1,\infty}(\mathbb{U}^{\alpha\beta}) \subset \mathcal{W}^{1,p}(\mathbb{U}^{\alpha\beta})$  and  $h_3^{\alpha\beta} = h_2$  on  $\partial\mathbb{U}^{\alpha\beta}$ , so we have

$$h_3^{\alpha\beta} - h_2 \in \mathcal{W}_\circ^{1,p}(\mathbb{U}^{\alpha\beta}).$$

**Step 3c.** We now define a homeomorphism  $h_3: \mathbb{X} \xrightarrow{\text{ontg}} \mathbb{Y}$  by the rule

$$h_3 = \begin{cases} h_3^{\alpha\beta} & \text{in } \mathbb{U}^{\alpha\beta} \\ h_2 & \text{in } \mathbb{X} \setminus \bigcup_{\alpha\beta} \mathbb{U}^{\alpha\beta}. \end{cases}$$

Obviously,  $h_3$  is a  $\mathcal{C}^\infty$ -diffeomorphism in  $\mathbb{X}_\circ$  and  $h_3 - h_2 \in \mathcal{W}_\circ^{1,p}(\mathbb{X}_\circ)$ . Since  $h_3$  coincides with  $h_2$  outside  $\mathbb{X}_\circ$  we have  $h_3 = h_2 + [h_3 - h_2]_\circ$ . Hence

$$(A_3) \quad h_3 - h_2 \in \mathcal{A}_\circ(\mathbb{X}).$$

Then, for every  $x \in \mathbb{X}$ ,

$$|h_3(x) - h_2(x)| \leq \begin{cases} \text{diam } h_2(\mathbb{U}^{\alpha\beta}), & \text{for } x \in \mathbb{U}^{\alpha\beta} \\ 0, & \text{otherwise} \end{cases} \leq \text{diam } \Upsilon^\alpha \leq \frac{\epsilon}{5}$$

meaning that

$$(B_3) \quad \|h_3 - h_2\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$



The computation of  $p$ -norms goes as follows

$$\begin{aligned}
 \|\nabla h_3 - \nabla h_2\|_{\mathcal{L}^p(X)}^p &= \sum_{\alpha\beta} \int_{\mathbb{V}^{\alpha\beta}} |\nabla h_3 - \nabla h_2|^p \\
 &\leq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \left[ \|\nabla h_3\|_{\mathcal{C}(\mathbb{V}^{\alpha\beta})} + \|\nabla h_2\|_{\mathcal{C}(\mathbb{V}^{\alpha\beta})} \right]^p \\
 &\leq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| (m_{\alpha\beta} + M_{\alpha\beta})^p \leq \sum_{\alpha\beta} \frac{\epsilon^p}{5^p 2^{\alpha+\beta}} \leq \left(\frac{\epsilon}{5}\right)^p.
 \end{aligned}$$

Hence

$$(C_3) \quad \|\nabla h_3 - \nabla h_2\|_{\mathcal{L}^p(X)} \leq \frac{\epsilon}{5}.$$

In the finite energy case, when  $\|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})} < \infty$ , we observe that

$$\|\nabla h_3\|_{\mathcal{L}^p(\mathbb{X} \setminus \bigcup \mathbb{V}^{\alpha\beta})} = \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X} \setminus \bigcup \mathbb{V}^{\alpha\beta})} \leq \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})}.$$

Therefore, by triangle inequality,

$$\begin{aligned}
 \|\nabla h_3\|_{\mathcal{L}^p(\mathbb{X})} &\leq \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \|\nabla h_3\|_{\mathcal{L}^p(\mathbb{V}^{\alpha\beta})} \\
 &\leq \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \cdot \|\nabla h_3\|_{\mathcal{C}(\mathbb{V}^{\alpha\beta})} \\
 &\leq \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}} \cdot M_{\alpha\beta}
 \end{aligned}$$

which yields

$$(D_3) \quad \|\nabla h_3\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla h_2\|_{\mathcal{L}^p(\mathbb{X})} + \delta.$$

The third step is completed.

#### STEP 4

We have already upgraded the mapping  $h$  to a homeomorphism  $h_3: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$  that is a  $\mathcal{C}^\infty$ -diffeomorphism in  $\mathbb{X}_o = \mathbb{X} \setminus \bigcup_{v \in \mathcal{V}(\mathbb{X})} \overline{\mathbb{F}_v}$ , where  $\mathbb{F}_v$  are small surroundings of the vertices of cells. Their images  $h_3(\mathbb{F}_v) = h_2(\mathbb{F}_v) = \mathbb{D}_c$  are small disks centered at  $c = h(v)$ . In Step 3a, one of the preconditions on those disks was that  $\text{diam } \mathbb{D}_c < \epsilon/5$ . Furthermore, the closed disks  $\overline{\mathbb{D}_c}$  are isolated continua in  $\mathbb{Y}$  for all  $c \in \mathcal{C}(\mathbb{Y})$ , so are the sets  $\overline{\mathbb{F}_v}$  in  $\mathbb{X}$ . We shall now consider slightly larger concentric open disks  $\mathbb{D}'_c \supset \overline{\mathbb{D}_c}$ ,  $c \in \mathcal{C}(\mathbb{Y})$ , and their preimages  $\mathbb{F}'_v = h_3^{-1}(\mathbb{D}'_c) \subset \mathbb{X}$ ,  $v = h^{-1}(c) \in \mathcal{V}(\mathbb{X})$ . The annulus  $\mathbb{D}'_c \setminus \overline{\mathbb{D}_c}$  will be thin enough to ensure that  $\mathbb{D}'_c$  are still disjoint,

$$\text{diam } \mathbb{D}'_c < \frac{\epsilon}{5} \quad \text{for all } c \in \mathcal{C}(\mathbb{Y})$$

and

$$\sum_{v \in \mathcal{V}(\mathbb{X})} \|\nabla h_3\|_{\mathcal{L}^p(\mathbb{F}'_v \setminus \mathbb{F}_v)}^p \leq \left(\frac{\epsilon}{20}\right)^p.$$

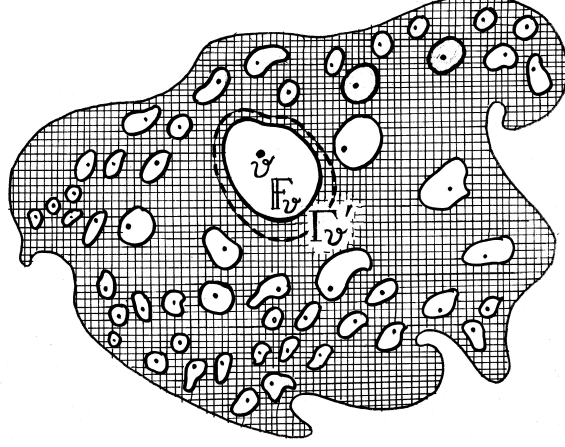


FIGURE 5. Neighborhoods of vertices.

Let  $\Gamma'_v$ ,  $v \in \mathcal{V}(\mathbb{X})$ , denote the boundary of  $\mathbb{F}'_v$ . These are  $\mathcal{C}^\infty$ -smooth Jordan curves. We now define a homeomorphism  $\tilde{h}_4: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$  by performing  $p$ -harmonic replacement of mappings  $\tilde{h}_3: \mathbb{F}'_v \xrightarrow{\text{ontq}} \mathbb{D}'_c$ , whenever such a mapping fails to be  $\mathcal{C}^\infty$ -diffeomorphism. Thus every  $\tilde{h}_4: \mathbb{F}'_v \xrightarrow{\text{ontq}} \mathbb{D}'_c$  is a  $\mathcal{C}^\infty$ -diffeomorphism up to  $\Gamma'_v$ . Moreover  $\tilde{h}_4 \in \tilde{h}_3 + \mathcal{W}_o^{1,p}(\mathbb{F}'_c)$ , so

$$(A_4) \quad \tilde{h}_4 - \tilde{h}_3 \in \mathcal{A}_o(\mathbb{X}).$$

For every  $x \in \mathbb{X}$ , we have

$$|\tilde{h}_4(x) - \tilde{h}_3(x)| \leq \begin{cases} \text{diam } \mathbb{D}'_c & \text{in } \mathbb{F}'_v, c = h(v) \\ 0 & \text{otherwise} \end{cases} \leq \frac{\epsilon}{5}.$$

Hence

$$(B_4) \quad \|\tilde{h}_4 - \tilde{h}_3\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

By virtue of the minimum energy principle we compute the  $p$ -norms

$$\begin{aligned}
 \|h_4 - h_3\|_{\mathcal{L}^p(\mathbb{X})}^p &= \sum_{v \in \mathcal{V}(\mathbb{X})} \|h_4 - h_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}^p \\
 &\leq \sum_{v \in \mathcal{V}(\mathbb{X})} [\|h_4\|_{\mathcal{L}^p(\mathbb{F}'_v)} + \|h_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}]^p \\
 &\leq 2^p \sum_{v \in \mathcal{V}(\mathbb{X})} \|h_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}^p \\
 &\leq 2^{2p-1} \sum_{v \in \mathcal{V}(\mathbb{X})} [\|h_3\|_{\mathcal{L}^p(\mathbb{F}'_v \setminus \mathbb{F}_v)}^p + \|h_3\|_{\mathcal{L}^p(\mathbb{F}_v)}^p] \\
 &\leq 2^{2p-1} \left[ \left( \frac{\epsilon}{20} \right)^p + \sum_{v \in \mathcal{V}(\mathbb{X})} \|h_2\|_{\mathcal{L}^p(\mathbb{F}_v)}^p \right] \\
 &\leq 2^{2p} \left( \frac{\epsilon}{20} \right)^p = \left( \frac{\epsilon}{5} \right)^p.
 \end{aligned}$$

Hence

$$(C_4) \quad \|h_4 - h_3\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

Again by minimum energy principle we find that

$$(D_4) \quad \|h_4\|_{\mathcal{L}^p(\mathbb{X})}^p \leq \|h_3\|_{\mathcal{L}^p(\mathbb{X})}^p.$$

Just as in the previous steps, condition  $(E_4)$  remains valid, finishing Step 4.

#### STEP 5

The final step consists of smoothing  $h_4$  in a neighborhood of each smooth Jordan curve  $\Gamma'_v$ ,  $v \in \mathcal{V}(\mathbb{X})$ . We argue in much the same way as in Step 3, but this time we appeal to Proposition 2.9 instead of Proposition 2.8. By smoothing  $h_4$  in a sufficiently thin neighborhood of each  $\Gamma'_v$  we obtain a  $\mathcal{C}^\infty$ -diffeomorphism  $h_5: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ,

$$(A_5) \quad h_5 - h_4 \in \mathcal{A}_0(\mathbb{X}).$$

$$(B_5) \quad \|h_5 - h_4\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

$$(C_5) \quad \|h_5 - h_4\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

$$(D_5) \quad \|h_5\|_{\mathcal{L}^p(\mathbb{X})} \leq \|h_4\|_{\mathcal{L}^p(\mathbb{X})} + \delta. \quad \square$$

## 4. OPEN QUESTIONS

**Question 4.1.** Does Theorem 1.1 extend to  $n = 3$ ?

**Question 4.2.** A bi-Sobolev homeomorphism  $h: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$  is a mapping of class  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ ,  $1 \leq p < \infty$ , whose inverse  $h^{-1}: \mathbb{Y} \xrightarrow{\text{ontq}} \mathbb{X}$  belongs to a Sobolev class  $\mathcal{W}^{1,q}(\mathbb{Y}, \mathbb{X})$ ,  $1 \leq q < \infty$ . Can  $h$  be approximated by bi-Sobolev diffeomorphisms  $\{h_\ell\}$  so that  $h_\ell \rightarrow h$  in  $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$  and  $h_\ell^{-1} \rightarrow h^{-1}$  in  $\mathcal{W}^{1,q}(\mathbb{Y}, \mathbb{X})$ ?

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